# A Few Remarks on Geometry and Physics

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Non-commutative algebra began in 1843 with William Rowan Hamilton's discovery that rotations in 3-d space are non-commutative. This means that the order in which one performs rotations in 3-d affects the outcome. In 3-d space there are 3 rotational degrees of freedom. In aviation these are called roll, pitch and yaw. Imagine that you are piloting an airplane which is in a level flight headed due west. If you roll to the right 90 degrees, and then pitch forward by 90 degrees, you will then be headed straight south with your right wing pointed straight down. Now if you had performed these rotations in the opposite order, you would first have pitched forward by 90 degrees, and then rolled to the right 90 degrees. In this case you would be headed straight down with your right wing pointing east. A big difference!

The reason it took Hamilton (a brilliant and imaginative mathematician and physicist) 15 years to discover the first non-commutative algebra (which he called quaternions), is that roations in 2-d space are commutative. For example:

$$30 \text{ deg.} + 60 \text{ deg.} = 60 \text{ deg.} + 30 \text{ deg.}$$

For someone who doesn't fly an acrobatic airplane, the non-commutativity of 3-d rotations is very counter-intuitive.

There is a subtle point that needs to be made here. Although the 3-d rotations are non-commutative, the space on which the rotations act (i.e., ordinary Euclidean 3-d space) has a commutative basis (which is usually written as the Cartesian coordinates basis of x, y, z). However, the algebra of 3-d rotations (i.e., quaternions) is a non-commutative structure (and as a 4-d vector space has the non-commutative basis of 1, i, j, k, where according to Hamilton's famous formula:

$$i^2 = j^2 = k^2 = ijk = -1$$

Now the full 4-d quaternion algebra is capable of doing much more than rotations in 3-d space. It can also do rotations in 4-d space, as well as stretchings (and shrinkings) of 3-d vectors. So, usually, the rotations of 3-d space are handled by a substructure of the quaternions which is called SO(3), the set of special orthogonal 3-by-3 matrices. This is a non-commutative Lie group.

Note: an algebra is an analog to the ordinary algebra of (real or complex) numbers in which both addition and multiplication can be performed. A linear algebra (such as quaternion algebra) is a vector space (for the additive part) and a multiplicative structure defined on this vector space. The multiplicative structure (such as matrix multiplication) contains many elements with no multiplicative inverses. If we collect together all the elements of the algebra which have multiplicative inverses, then we have a Lie group (in general a non-commutative Lie group. [A Lie group is a group which is also a space.] The additive structure of the algebra is itself a commutative Lie group.

So, in general, a linear algebra consists of:
(1) a vector space, which is an additive (commutative) Lie group; and
(2)a multiplicative structure, which includes both invertible and non invertible elements;
(3) an embedded, multiplicative Lie group made up of the invertible elements.
[These multiplicative structures are usually non-commutative. In fact, the only linear algebras with commutative multiplicative structures are the real numbers R, and the complex numbers C, so that these are called "fields"--not to be confused with the use of the word "field" to refer to an assignment of some mathematical entity (such as a vector) to each point of some manifold.]

It is important to realize that, although the linear algebra is a vector space, the imbedded multiplicative Lie group is not a vector space; and thus, as a space, the multiplicative Lie group (usually) possesses curvature, although it is imbedded in a flat space, which is the vector space of the algebra.

[Note on Torsion: A noncommutative Lie group also posses torsion; and the Lie group torsion tensor is the structure constant tensor of the Lie algebra of left-invariant vector fields of the Lie group. But this Lie algebra is (usually) different from the Lie algebra in which the Lie group is imbedded as the set of invertible elements. The only cases in which these two Lie algebras are the same are the total matrix algebras. If we consider the set of all n-by-n matrices, then these form the total matrix algebra M(n), and this is an associative algebra which can be made into a Lie algebra by defining the Lie product commutator on the underlying associative algebra. In this case the set of all invertible elements form a Lie group called the General Linear group, GL(n); and the Lie algebra of left-invariant vector fields of GL(n) is the same as the Lie algebra of M(n), and is called gl(n).]

The relationship of SO(3) to quaternion algebra is as follows:
The quaternion algebra is a 4-d vector space, in which the set of all unit-length vectors constitute a 3-d sphere (just like the unit length vectors in 3-d space form a 2-d sphere). This 3-d sphere is a Lie group called SU(2) --i.e., the set of all special unitary 2-by-2 matrices. There is a 2-to-1 mapping of SU(2) to SO(3). This means that for each pair of positive and negative elements (g and -g) of SU(2) there is a single element of SO(3). So we can symbolize this by writing SO(3) = SU(2) / {+1,-1}.

Physicists usually learn about this 2-to-1 mapping when they deal with particles of half-integral spin such as electrons and protons. The state of such a particle, when rotated through 360 degrees changes its sign from positve to negative; and when rotated a further 360 degrees, returns to its original form. It can be visualized by P.A.M. Dirac's famous trick in which an object (such as a ball) is connected by rubber strings to the floor and ceiling of a room (a minimum of 3 strings will do). When the ball is rotated through 360 degrees the rubber strings are twisted up in such a way that they cannot be untwisted. However, when the ball is rotated through a further 360 degrees (in the same direction as before), the rubber strings can be untwisted and returned to their original positions.

In this discussion of algebra and geometry there are several issues which speak to Joe Firmage's questions: Cartesian-friendlyness, dimensionality, curvature, pictureability, and mechanism.

(1) Linear algebra is very Cartesian-friendly. I assume that "Cartesian" refers to the flatness of the 3-d vector space in which Rene Descartes erected his coordinate system (x,y,z) in making a correspondence between algebraic expressions (equations in x, y, z) and lines in 3-d space. A linear algebra, by definition, is a (flat) vector space, which acts on a variety of vector spaces (including itself as a vector space). A linear algebra is represented by matrices--i.e., each element of the algebra corresponds to a matrix. And a matrix acts on a vector by changing it into a different vector. A d-by-d matrix acts on a vector-space of dimension d. In this action, the vector may be rotated, or stretched, or both. The matrix algebra of all d-by-d matrices is itself a vector space of $d^2$ dimensionality, because there are $d^2$ little boxes in a d-by-d matrix. The box-by-box additivity of matrices corresponds to the vector-space structure of a matrix algebra.

(2) Dimensionality: Spaces of more than 3-d were a natural outgrowth of Descartes' mapping between polynomials and lines in "Cartesian" space. Although the lines drawn in 3-d space correspond to polynomials in x, y, and z, the Cartesian correspondence raises the natural question: What if we try to describe (geometrically) polynomials in more than three variables? Although Descartes published his "Cartesian" geometry (now called analytic geometry) in 1737, it was not until 1843 that Arthur Cayley published in the Cambridge Mathematical Journal the paper, "Chapters in the Analytical Geometry of n Dimensions." Cayley also introduced the word "hyperspace" in 1867 in the phrase, "the quasi-geometrical representation of conditions by means of loci in hyperspace," published in *Mathematical Papers* vol. VI, p. 191. (1893).

(3) Curvature: It was Bernhard Riemann (as a graduate student of Karl Friedrich Gauss) who in 1854 combined both hyperspace and non-Euclidean (curved) geometry in the famous lecture "On the Hypotheses which Lie at the Foundations of Geometry." Gauss had already described the curvature of 2-d surfaces (imbedded in 3-d space) in such a way that the curvature can be considered independent of the imbedding. Riemann extended this idea to any number of dimensions as "an n-fold extended quantity." This lecture was not published until 1867.

It should be noted that Riemann defined an n-dimensional manifold in such a way that it is locally just like an n-dimensional Euclidean space. Thus Riemann's manifolds, which can be curved (and of any dimension) retain a basic relationship to Cartesian structures.

Moreover, William Clifford (1845-1879 translated Riemann's work into English, and under Riemann's influence proposed that the curvature of space varying in time corresponds to the phenomena of the physical world. This speculation was published in 1885 after his death at age 34 in the book *Common Sense of the Exact Sciences.*

Albert Einstein's theory of general relativity (1915) draws heavily on the work of Riemann; and Clifford's idea is partially fulfilled in the sense that gravity is accounted for as the curvature (i.e., the degree of departure from Euclidean flatness) of space-time. Clifford wanted to account for all of physics by the change of space curvature as time goes on. The idea of space-time curvature had to wait for Einstein's special relativity (1905) and Herman Minkowski's (1908) unification of space and time into a single 4-d (flat) manifold--now called "Minkowski-space." Einstein at first thought of minkowski-space as a "superfluous sophistication," but later used the curvature of 4-d space-time to provide a model of gravitation.

In order to fulfill Clifford's vision it seems necessary to combine spacetime with an extra dimension into a curved 5-d spacetime. This was done by Theodore Kaluza (1921) and Oscar Klein (1926). This allows one to account for both gravity and electro-magnetism (in the sense of Maxwell's equations). In this case, particles feeling both gravity and electromagnetism move along geodesics in the curved 5-d space. However, as Klein discovered, in order for the charge of the electron to be properly related to the strength of gravity, the extra dimension must be sub-microscopic--i.e., of the order of the Planck length in radius -- $10^{-33} cm$.

The Kaluza-Klein unification program is to enlarge the dimensionality of space-time enough so that all physical forces (including the strong and weak nuclear forces) can be brought into the geodesic picture. The question then is: Given an N-dimensional curved space, at what N do all the particles move along geodesics (and no other paths) in the hyperspace, assuming that all forces are active.

Already, the Planck length brings in quantum theory, so that the K-K program must incorporate both general relativity and quantum mechanics. This has turned out to be the most difficult issue.

In a sense, the string theory program (which is somewhat different) is a fulfillment of the Kaluza-Klein program. The first string theory of 1969 was designed to deal only with the strong nuclear force. It was a total embarrassment because: (1) it could deal only with bosons (integral spin particles) such as mesons, and not with fermions (half-integral spin particles) such as baryons (e.g., protons and neutrons). (2) It would work only if space-time was 26 dimensional. (3) The basic particle would have to be a massless spin-2 particle.

In the early 1970s, supersymmetry as invented to deal with the first problem. Supersymmetry transforms bosons into fermions and vice versa. This had two good consequences. (1) Space-time is reduced from 26 to 10 dimensions. (2) Also supersymmetry transformations, when allowed to vary over spacetime have an effect similar to gravity (this is called supergravity, and supergravity theory evolved into a point-particle 11-dimensional theory).

Most importantly, the massless spin-2 particle can be recognized as the long hypothesized graviton (whose spin-2 quality is necessary to keep gravity always attractive.)

[Now if only these spin-2 gravitons could be considered as composites of spin-1 particles (because spin is an additive quantum number), one might be able to model an anti-gravity scheme...but that is another story.]

Thus in 1974, John Schwarz and Joel Scherk proposed the idea that superstring theory (10-d) should be considered not primarily as a strong-force theory, but as a quantum gravity theory. It took 10 years for this idea to take hold; but, in 1984, when John Schwarz and Michael Green proved that superstring theory is consistent ("renormalizalbe, anomaly free") quantum gravity theory, many physicists suddenly became interested in superstring theory.

It is important to realize that, so far, superstring theory (which includes supergravity) is the only consistent quantum gravity theory.

Moreover, with an eye on the K-K program, it is possible to view superstring theory as a unification of all the forces in quantum-mechanical viable manner.

There have been many advances in superstring theory since 1984. For 10 years there were five competing superstring theories (two of them, called "heterotic" actually subtle weavings together of the 10 and 26 dimensional theories). By 1995, through the work of Edward Witten (and many others) these five competing theories have been shown to be subtheories in an overarching theory called M-theory ("membrane, mystery, or magic according to taste"--as Witten says). The big surprise here, is that there is a kind of master theory, 11-d supergravity theory. It should be noted that supergravity theory is not a string theory, but a particle theory. However, in the context of M-theory, membranes can be considered of any dimension between: 0 (particles), 1 (strings), 2 (ordinary membranes), 3,...,9.

Moreover, the 7 hidden dimensions of 11-d supergravity is accounted for by the 7-d torus (the maximal commutative subgroup) of the $E_7$ Lie group. When the hidden dimensions are toroidal, each dimension can be of a different size. They don't all have to be around the Planck-length in radius. Thus there is some hope of "seeing" one or more of these hidden dimensions in particle accelerator experiments of the the next couple of decades.

In fact there is indirect evidence for supersymmetry. And there is the distinct possibility that supersymmetry partners of ordinary particles have already been seen, or will soon be seen in accelerator experiments at the Fermi NAL or at CERN when the new accelerator goes on line around 2005.

(4) Picturability and mechanism:

If we are ever to tap zero-point energy, we must understand the basic structure of space. The standard picture of vacuum fluctuations is that of John Wheeler. In his many pictures, he shows that when we look at small enough regions of spacetime ( approaching the Planck scale), the fluctuations of spacetime become so violent that it breaks up into a "quantum foam." The string theory picture of the realm of the Planck scale is radically different. In this picture, when we approach the Planck scale, we begin to see the fluctuations of the strings, and spacetime becomes 10 dimensional. Thus there are 8 transverse dimensions for the fluctuations to take place, in addition to the 2 dimensions of the world sheet swept out by the movement of the 1-dimensional strings (analogous to world lines swept out by point particles). Moreover, the fluctuations are harmonic, and the harmonic modes correspond to particle states.

[Incidentally, the strings are also spinning, so that there is torsion in superstring theory. Moreover, this is propagating torsion. And this needs to be looked into.]

Are these fluctuations zero-point energy fluctuations? Yes! In fact, the 10 dimensions of the superstring spacetime can be derived from a zero-point energy calculation on the strings. This was done in 1973 by Lars Brink and H.B. Nielsen. Their paper, “A Simple Physical Interpretation of the Critical Dimension of Space-Time in Dual Models,” Physics Letters, Vol. 45B, number 4, pp. 332-336 (6 August 1973). This paper is included in the anthology edited by John H. Schwarz, Superstrings: the first 15 years of superstring theory, Volume 1 (World Scientific, 1985).

The basic idea here is stated as: “The ground state of the quantum mechanical string must have an energy due to zero-point fluctuations of the harmonic oscillators into which the dynamical variables of the the string can be expanded.”

I would say that this is a very beautiful picture; and a very beautiful mechanism which has a close resemblance to the harmonics of music--the music of the strings!