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Non Holonomic Constraints in Newtonian Mechanics
Provides a Picture of Post-Bohmian Back-Action.

Pedagogical Review from the Classics of Physics

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Section 5 of Vigier 2000.pdf

Second draft-under construction

My footnoted commentaries to Sommerfeld are especially important to read.

Before we get lost in formalism, a brief return to physics: freely adapted from Arnold Sommerfeld's¹ "Mechanics". Ch II "Mechanics of systems, Principle of Virtual Work and D'Alembert's Principle, #7 Degrees of Freedom and Virtual Displacements; Holonomic and Non-holonomic constraints"

The number of degrees of freedom for n mass points which are coupled by r relations between their coordinates is

$$f = 3n - r \quad (5.1)$$

The freely moving rigid body has

$$f = 3 + 2 + 1 = 6 \quad (5.2)$$

Any point on the body has 3, a second point a fixed distance from first has 2, finally a third point can circle the axis formed by the first two points so 1 more.

Spinning top on a plane: bottom of top terminates at a point on the plane. This point on the plane has two translational degrees of freedom. A second point can move on a hemisphere about the first, and a third one on a circle about a line connecting the first two.

$$f = 2 + 2 + 1 = 5 \quad (5.3)$$

¹ Arnold Sommerfeld's students in Munich included Werner Heisenberg, Peter Debye, Wolfgang Pauli, and Hans Bethe. I was fortunate to have Hans Bethe as a one on one tutor at Cornell in 1960 and in a small seminar of maybe six seniors. Bethe spoke often of Sommerfeld, who also worked with the great mathematician Felix Klein in Gottingen. Bethe was noted for his elegant clarity that he inherited from Sommerfeld.

Top with fixed point on plane

$$f = 2 + 1 = 3 \quad (5.4)$$

For a deformable solid body, liquid and field

$$f = \infty \quad (5.5)$$

and the equations of motion are partial differential equations rather than a finite system of ordinary differential equations of second order.

The number of degrees of freedom equals the number of independent coordinates which are necessary to determine the configuration of the mechanical system. These coordinates need not be Cartesian. For example, the drive mechanism of piston engine

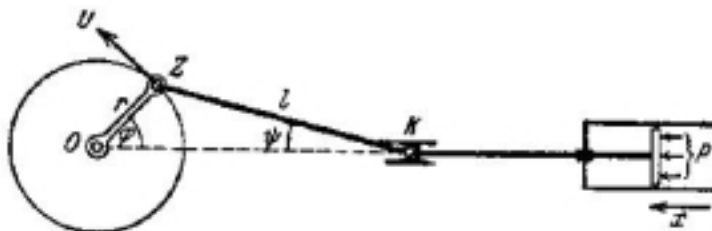


FIG. 9. Schematic diagram of the drive mechanism of a piston engine.

we can equally well specify either the coordinate x determining the position of the piston or the angle φ giving the position of the crank pin on the shaft. In general call the independent coordinates of a system of f degrees of freedom

$$q_1, q_2, \dots, q_f \quad (5.6)$$

They can, within certain limits, be chosen arbitrarily. The r conditions among the coordinates in eq (5.1) can be satisfied identically by a suitable choice of the q , so that they drop out of the subsequent description of the system.

Hertz called attention to *conditions of differential form* to which the foregoing cannot be applied. Such a condition is

$$\sum_{k=1}^f F_k(q_1, q_2, \dots, q_f) dq_k = 0 \quad (5.7)$$

Here we assume that the F_k do not all have the form $\frac{\partial\Phi}{\partial q_k}$, so that (5.7) is *not* the *total differential* of some function $\Phi(q_1, q_2, \dots, q_f)$ and we assume, moreover, that it cannot be converted into a total differential by means of an integrating factor². Following Hertz, conditions of the form

$$\Phi(q_1, q_2, \dots, q_f) = \text{const} \quad (5.8)$$

are called *holonomic* (*holos* in Greek = integer in Latin = whole = integrable). In contrast, conditions of the form (5.7), which cannot be formally integrated, will be called *non-holonomic* or *anholonomic*. That is, at for at least some of the F_k

$$F_k \neq \frac{\partial\Phi}{\partial q_k} \quad (5.9)$$

The simplest example of a non-holonomic constraint is a sharp-edged wheel of radius R rolling without sliding on a rough horizontal plane support. We need to distinguish finite degrees of freedom from infinitesimal degrees of freedom. The instantaneous position of the wheel has two translational degrees of freedom x, y of the point of contact of the wheel with the plane. The angle ψ formed between the tangent to the wheel (intersection of the plane of the wheel with that of the support plane) and the x -axis. The angle of rotation ϕ about the axle of the wheel of the point of contact of the wheel with the plane (odometer reading). Finally, the angle of tipping θ that the axle makes with the normal to the x - y plane if it is not constrained to be a vertical wheel. Therefore, there are 5 finite degrees of freedom. The mobility of the wheel is, however, restricted by the condition of pure rolling (without slipping) brought about by the static friction between the wheel and support; indeed, with the wheel moving along its instantaneous direction, the distance δs moved along the direction of the tangent must equal $R\delta\phi$. Bt projecting on the coordinate axes we then obtain the conditions of constraint

$$\delta x = R \cos \psi \delta \phi, \delta y = R \sin \psi \delta \phi \quad (5.10)$$

Hence the rolling slanting wheel has only three infinitesimal degrees of freedom. The conditions (5.10) cannot be reduced to equations between the 5 finite coordinates x, y, θ, ψ, ϕ themselves. Assuming the wheel is always vertical, θ is an ignorable constant of the motion. So the point here is that the existence of the equation

$$f(x, y, \phi, \psi) = 0 \quad (5.11)$$

is incompatible with (5.10).

² A multiplier used on both sides of the equation to simplify and solve differential equations.

$$\delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial \phi} \delta \phi + \frac{\partial f}{\partial \psi} \delta \psi \quad (5.12)$$

Substitute the infinitesimal constraints (5.10) into the condition for the total differential

$$\delta f = \left(\frac{\partial f}{\partial x} R \cos \psi + \frac{\partial f}{\partial y} R \sin \psi + \frac{\partial f}{\partial \phi} \right) \delta \phi + \frac{\partial f}{\partial \psi} \delta \psi \quad (5.13)$$

However, $\delta \phi, \delta \psi$ are independent variations. (5.11) implies $\delta f = 0$, therefore

$$\frac{\partial f}{\partial \psi} = 0 \quad (5.14)$$

and

$$R \frac{\partial f}{\partial x} \cos \psi + R \frac{\partial f}{\partial y} \sin \psi + \frac{\partial f}{\partial \phi} = 0 \quad (5.15)$$

Differentiate (5.15) by ψ and use (5.14), interchange the orders of $\frac{\partial}{\partial x}, \frac{\partial}{\partial \psi}$ etc. to get

$$-\frac{\partial f}{\partial x} \sin \psi + \frac{\partial f}{\partial y} \cos \psi = 0 \quad (5.16D)$$

Differentiate by ψ again to get

$$\frac{\partial f}{\partial x} \cos \psi + \frac{\partial f}{\partial y} \sin \psi = 0 \quad (5.17)$$

Therefore,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \phi} = 0 \quad (5.18)$$

And from (5.14) we see there is no non-constant function possible.

Similar examples are sleighs and the flexible couplings of bicycles.³

Such a wheel is locally restricted to move always in the direction it has at a given instant. Nevertheless it is able to reach all points of the plane, even if at times only pivoting about its sharp point of contact. This pivoting means that there is a discontinuity of the tangent line at the pivot point. This is also a simple kind of crinkle. It is a simple kind of Rene

³ and probably the Shipov machine and Dean Drive.

Thom catastrophe or V.I. Arnold singularity. Therefore, in a non-holonomic situation, a system with f finite global degrees of freedom has only $f-r$ infinitesimal local degrees of freedom.

Consider the Bohm point constrained to move inside its quantum potential landscape hypersurface threading through n -dimensional configuration space. Bohm showed that his "causal theory" covers orthodox quantum theory and it corresponds to the landscape being a holonomic constraint or gradient flow. The lack of back-action, needed to maintain the statistical predictions of quantum theory with absolute local randomness and signal-locality, means no post-quantum pivoting is allowed! Post-quantum theory is the non-holonomic generalization of holonomic quantum theory as seen only in the Bohm ontology.

To recapitulate:

"In general, if a system subject to r non-holonomic conditions has f degrees of freedom in finite motion, it has only $f - r$ degrees of freedom in infinitesimal motion. ... The foregoing distinction is important for the concept of virtual displacements. A virtual displacement is an arbitrary, instantaneous, infinitesimal change of the position of the system compatible with the conditions of constraint. Whereas we shall denote real displacements due to given forces under given conditions by

$$dq_1, dq_2 \dots dq_f \tag{5.19}$$

the symbols

$$\delta q_1, \delta q_2 \dots \delta q_f \tag{5.20}$$

will be used to denote virtual displacements. The δq have nothing to do with the actual motion. They are introduced, so to speak, as test quantities. Whose function is to make the system reveal something about its internal connections and about the forces acting on it." Sommerfeld's "Mechanics".

There is an analogy between virtual versus real displacements in classical mechanics and virtual and real photons in quantum electrodynamics. This analogy cannot be pushed too far of course. Virtual near field photons of longitudinal spatial polarization⁴ not seen in the far radiation field form the static field⁵ and velocity dependent forces that make the near field point toward the actual instantaneous position of the source charge relative to the test charge to a good approximation when $v/c \ll 1$. So, the virtual photons, like the virtual displacements, "reveal something about its internal connections and about the forces acting on it" since virtual photons are also internal lines in Feynman diagrams for perturbation theory. Local gauge invariance is a kind of non-holonomic constraint so that the two types of near field virtual photons are not independent of each other if total

⁴ The additional timelike AKA scalar polarization is not independent due to local gauge invariance.

⁵ The virtual photons are in Glauber coherent quantum states of optimized phase and number, i.e. minimum uncertainty wave packets.

electric charge is to be conserved. That is, we understand conservation of charge as an application of the principle of non-holonomic constraint.

"For purely holonomic constraints the δq are independent of each other, one δq corresponding to each degree of freedom. A larger number of δq must be introduced for non-holonomic constraints; in that case the δq are related by⁶

$$\sum_{k=1}^f F_k(q_1, q_2, \dots, q_f) \delta q_k = 0 \quad (5.21)^7$$

Here f is the number of degrees of freedom for finite motion. As previously emphasized, this number is greater than that for infinitesimal motion."⁸

⁶ Not by (5.8).

⁷ Do not confuse this equation (5.21) for virtual displacements with (5.6) for real displacements under the action of actual forces.

⁸ Therefore, (5.9) applies. We no longer have exact total differentials that only work for holonomic constraints with independent virtual displacements. Zero memory path independence of the histories break down. There is a natural hysteresis "memory" built into the anholonomic constraint shift from the exact unitary holonomic displacement of Bohm's gradient quantum potential flow to the inexact nonunitary nonholonomic post-quantum displacements in configuration space.